Received: *October 19, 2001*, Accepted: *November 5, 2001*

Brane collisions in anti-de Sitter space

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Abstract: From the requirement of continuous matching of bulk metric around the point of brane collision we derive a conservation law for collisions of p-branes in (p+2)-dimensional space-time. This conservation law relates energy densities on the branes before and after the collision. Using this conservation law we are able to calculate the amount of matter produced in the collision of orbifold-fixed brane with a bulk brane in the "ekpyrotic/pyrotechnic type" models of brane cosmologies.

Keywords: Field Theories in Higher Dimensions, Extra Large Dimensions, p-branes, Cosmology of Theories beyond the SM.

JHEP11(2001)007

Contents

1. Introduction.

In several recently proposed models of brane cosmology [1]–[4] the creation of four-dimensional expanding Friedman-Robertson-Walker (FRW) universe is associated with collisions of branes in higher-dimensional space-time. For example, in ekpyrotic/pyrotechnic scenario [1, 2] the visible universe is represented by a negative/positive tension 3-brane sitting on a fixed point of Z_2 orbifold in five-dimensional space-time. The orbifold brane collides inelastically with a brane moving through the bulk. At the moment of collision some amount of relativistic matter (radiation) is produced on the visible brane at the cost of kinetic energy of the bulk brane. Initially flat Minkowsky universe residing on the visible brane starts to expand. Thus, the initial conditions for the hot FRW universe are generated during the brane collision. An immediate problem to be addressed in these scenarios is what is the energy density of radiation (or, equivalently, what is the initial temperature of the universe) produced on the visible brane? Another question is whether the visible brane expands or collapses right after the collision? Indeed, naively, if one deposits homogeneous energy distribution into initially static universe, it will start to collapse.

In what follows we find a conservation law for the brane collisions which provides answers to the above questions. This conservation law is a direct generalization of a conservation law found by Dray and 't Hooft [5] for the case of collisions of light-like shells in four-dimensional Schwarzschild space-time.

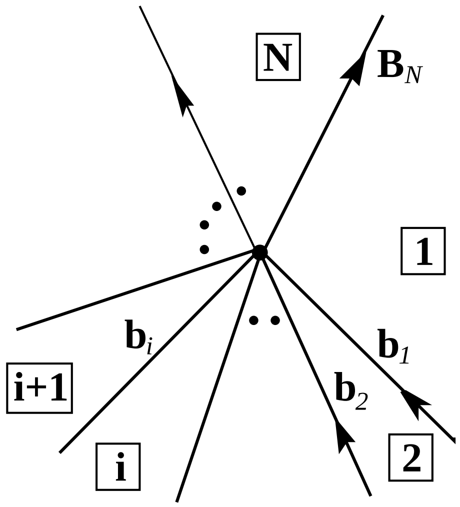

Figure 1: Colliding branes.

2. Conservation law in brane collisions.

Let us consider a collision of k in-branes $\mathbf{b}_1, \ldots, \mathbf{b}_k$ in which $N-k$ out-branes $\mathbf{B}_{k+1}, \ldots, \mathbf{B}_N$ are formed, as it is shown on figure 1. The N branes divide the space-time onto N disjoint regions **1,...,N**. Suppose that the whole space-time possesses a three-dimensional eucledian symmetry $E(3)$ so that the metric in each region **i** can be written in isotropic coordinates (U_i, V_i) as

$$ds^2 = -F_i(U_i, V_i) dU_i dV_i + H_i(U_i, V_i)\left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{2.1}$$

where F_i, H_i are some functions of (U_i, V_i) and (x_1, x_2, x_3) are coordinates of three-dimensional eucledian space. We can always rescale the coordinates x_k in such a way that $H_i = 1$ at the collision point.

If we want the space-time metric in the neighborhood of the collision point to be well-defined, we have to suppose that there exist isotropic coordinates (u, v) in which the space-time metric has the form

$$ds^2 = -f(u,v) du dv + h(u,v)\left(dx_1^2 + dx_2^2 + dx_3^2\right) \tag{2.2}$$

and the functions $f(u,v), h(u,v)$ are continuous at the collision point (u_c, v_c).

In each region **i** the coordinates (U_i, V_i) can be expressed through (u, v)

$$U_i = U_i(u), \qquad V_i = V_i(v). \tag{2.3}$$

Comparing the metrics (2.1) and (2.2) we find that the functions f, h are related to the functions F_i, H_i as

$$\begin{aligned} f(u,v) &= F_i\left(U_i(u), V_i(v)\right) U_i'(u) V_i'(v) \\ h(u,v) &= H_i\left(U_i(u), V_i(v)\right), \end{aligned} \tag{2.4}$$

where prime denotes the derivatives of functions (2.3). In particular, if we restrict (2.4) to the brane $\mathbf{b}_i$ we find

$$F_i(U_i,V_i)U_i'V_i'\big|_{\mathbf{b}_i} = F_{i+1}(U_{i+1},V_{i+1})U_{i+1}'V_{i+1}'\big|_{\mathbf{b}_i} \tag{2.5}$$

$$H_i(U_i,V_i)\big|_{\mathbf{b}_i} = H_{i+1}(U_{i+1},V_{i+1})\big|_{\mathbf{b}_i}\,. \tag{2.6}$$

The brane $\mathbf{b}_i$ which is the boundary between regions $\mathbf{i}$ and $\mathbf{i+1}$ is a surface defined by equation

$$\Sigma_i^+(U_i,V_i)=0 \tag{2.7}$$

in the coordinates of the region $\mathbf{i}$ or

$$\Sigma_{i+1}^-(U_{i+1},V_{i+1})=0 \tag{2.8}$$

in the coordinates of the region $\mathbf{i+1}$. If we take u as one of the coordinates along the brane $\mathbf{b}_i$ and take u derivative of (2.6) along $\mathbf{b}_i$ we find

$$\left(\partial_{U_i}H_i-\partial_{V_i}H_i\frac{\partial_{U_i}\Sigma_i^+}{\partial_{V_i}\Sigma_i^+}\right)U_i'=\left(\partial_{U_{i+1}}H_{i+1}-\partial_{V_{i+1}}H_{i+1}\frac{\partial_{U_{i+1}}\Sigma_{i+1}^-}{\partial_{V_{i+1}}\Sigma_{i+1}^-}\right)U_{i+1}'\,. \tag{2.9}$$

Taking (2.9) and (2.5) at the collision point (u_c,v_c) we find relations between U_i',V_i' and U_{i+1}',V_{i+1}'

$$\begin{pmatrix}U_{i+1}'\\ V_{i+1}'\end{pmatrix}=\mathcal{C}_{i(i+1)}\begin{pmatrix}U_i'\\ V_i'\end{pmatrix}, \tag{2.10}$$

where $\mathcal{C}_{i(i+1)}$ is a numerical 2×2 matrix. The transition matrices $\mathcal{C}_{12},\ldots,\mathcal{C}_{(N-1)N},\mathcal{C}_{N1}$ can be calculated for each pair of adjacent regions. An obvious consistency condition which must be satisfied if there exist coordinates (u,v) in the neighborhood of collision point such that the metric (2.2) is continuous at (u_c,v_c) is

$$\mathcal{C}_{12}\ldots\mathcal{C}_{(N-1)N}\mathcal{C}_{N1}=I \tag{2.11}$$

where I is the identity matrix. In what follows we show that this consistency condition provides us with a nontrivial conservation law, which relates the physical parameters of the in-branes $\mathbf{b}_1,\ldots,\mathbf{b}_k$ to the parameters of the out-branes $\mathbf{B}_{(k+1)},\ldots,\mathbf{B}_N$.

We consider the bulk metric of the form

$$ds^2=-F(R)dT^2+\frac{dR^2}{F(R)}+R^2\left(\sum_{k=1}^{3}dx_k^2\right), \tag{2.12}$$

where $F(R)$ is a given function. Substituting (2.12) in the bulk Einstein equations with a negative cosmological constant $-\Lambda$ we find a general solution

$$F(R)=\Lambda R^2-\frac{M}{R^2}, \tag{2.13}$$

where M is an integration constant. Let us write the condition (2.9) for this metric in more detailed form. In order to go to the isotropic coordinates (U, V) in the metric (2.12) we make a coordinate change

$$dz = -\frac{dR}{F(R)} \tag{2.14}$$

so that the metric becomes

$$ds^2 = F(R)\left(-dT^2 + dz^2\right) + R^2\left(\sum_{k=1}^{3} dx_k^2\right) . \tag{2.15}$$

The coordinates (U, V) are then

$$\begin{aligned} U &= T - \sigma z(R) \\ V &= T + \sigma z(R) , \end{aligned} \tag{2.16}$$

where $\sigma = +1$ if z grows to the right and $\sigma = -1$ is z grows to the left (if a space-time region **i** is a Z_2 image of a space-time region **j** with $\sigma_j = +1$ then $\sigma_i = -1$ and vise versa).

The trajectories of branes are given by the functions

$$\Sigma = R - \bar{R}(T) = 0 , \tag{2.17}$$

where $\bar{R}(T)$ is found from the equations of motion of the brane. Substituting (2.17) into the l.h.s. or r.h.s. of the matching condition (2.9) we find

$$\left(\partial_U H - \partial_V H \frac{\partial_U \Sigma}{\partial_V \Sigma}\right) = \frac{2RF\bar{R}'}{(F + \sigma\bar{R}')} . \tag{2.18}$$

Introducing the proper time along the brane through the relation

$$d\tau^2 = \left(F - \frac{\bar{R}'^{\,2}}{F}\right) dT^2 \tag{2.19}$$

we can rewrite (2.18) as

$$\left(\partial_U H - \partial_V H \frac{\partial_U \Sigma}{\partial_V \Sigma}\right) = \frac{2RF\dot{\bar{R}}}{\left(\dot{\bar{R}}^2 + F\right)^{1/2} + \sigma\dot{\bar{R}}} , \tag{2.20}$$

where dot denotes the proper time derivative. Taking the values of the last expression for regions **i** and **i+1** we find form (2.9)

$$U'_{i+1} = \left[\frac{F_i\left(\left(\dot{\bar{R}}^2 + F_{i+1}\right)^{1/2} + \sigma\dot{\bar{R}}\right)}{F_{i+1}\left(\left(\dot{\bar{R}}^2 + F_i\right)^{1/2} + \sigma\dot{\bar{R}}\right)}\right] U'_i \tag{2.21}$$

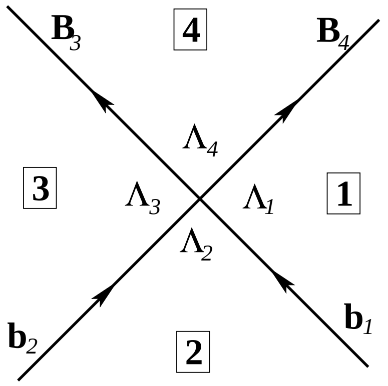

Figure 2: Collision of light-like branes.

if $\sigma_i = \sigma_{i+1} = \sigma$ or

$$U'_{i+1} = \left[\frac{\left(\dot{\bar{R}}^2 + F_i\right)^{1/2} - \sigma\dot{\bar{R}}}{\left(\dot{\bar{R}}^2 + F_i\right)^{1/2} + \sigma\dot{\bar{R}}} \right] U'_i \tag{2.22}$$

if $\sigma_i = -\sigma_{i+1} = \sigma$.

A simple example of brane collisions is presented on figures 2. The light-like branes $\mathbf{b}_1, \mathbf{b}_2, \mathbf{B}_3, \mathbf{B}_4$ are boundaries between the regions **1, 2, 3** and **4** which are all patches of anti-deSitter space-time with different cosmological constants $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$. In this case the consistency condition (2.11) gives (see appendix for details of calculation)

$$\Lambda_1\Lambda_3 = \Lambda_2\Lambda_4\,. \tag{2.23}$$

3. Collision of a bulk brane with an orbifold-fixed brane.

We proceed with a more complicated example in which a Z_2-orbifold brane $\mathbf{v}$ is hit inelastically by a bulk brane $\mathbf{b}$, as it is shown on figure 3. It is convenient to unfold Z_2 orbifold so that the space-time consists of four regions, the region **3** is a mirror copy of the region **2** and the region **4** is a mirror copy of the region **1**. The situation presented on figure 3 was discussed recently in the "ekpyrotic" and "pyrotechnic" scenarios of brane cosmology.

We can make different assumptions about the bulk metric in regions **1** and **2**. For example, we can suppose that the cosmological constants in these regions are different, Λ_1 and Λ_2 respectively. Otherwise we can suppose that the cosmological constants are the same, $\Lambda_1 = \Lambda_2 = \Lambda$, but the metric in region **1** is different from anti-deSitter

$$F_1 = \Lambda R^2 - \frac{M_1}{R^2} \tag{3.1}$$

we will consider both possibilities.

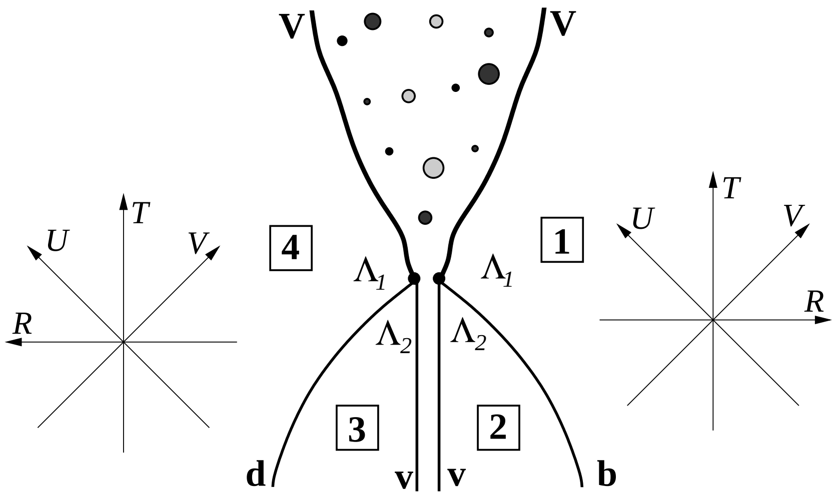

Figure 3: Collision of a bulk brane **b** with an orbifold brane **v**.

In order to write down the matching conditions (2.21) or (2.22) between different regions we need to specify the value of parameter σ in each region. If we take $\sigma = +1$ in the regions **1** and **2** this will automatically imply the $\sigma = -1$ in the regions **4** and **3** since these regions are mirror copies of **1, 2**. In this case the coordinate z (2.14) grows to the right in the region **1** while the coordinate R

$$R = \frac{1}{\Lambda z} \tag{3.2}$$

grows to the left. One can check that the tensions of the visible branes **v, V** are positive in this case. If we take $\sigma = -1$ in the regions **1, 2** the visible branes have negative tension. The latter situation corresponds to the one considered in the original ekpyrotic scenario [1] while the former has been analyzed in pyrotechnic model [2].

The trajectory of the bulk brane **b** is $\bar{R}_b(\tau)$. It is convenient to introduce the expansion rate

$$H_b = \frac{\dot{\bar{R}}_b}{\bar{R}_b}, \tag{3.3}$$

where dot denotes the proper time derivative. The matching condition (2.21) between the regions **1** and **2** reads

$$U_2' = \left[\frac{F_1(\sqrt{H_b^2 + F_2/R^2} + \sigma H_b)}{F_2(\sqrt{H_b^2 + F_1/R^2} + \sigma H_b)}\right] U_1' \, . \tag{3.4}$$

The trajectory of the visible brane **V** after the collision is $\hat{R}_V(\tau)$ and its expansion rate is H_V. Since the region **4** is a mirror copy of the region **1** we use the condition

(2.22) for matching the regions **1** and **4**

$$U_4' = \left[\frac{\sqrt{H_V^2 + F_1/R^2} - \sigma H_V}{\sqrt{H_V^2 + F_1/R^2} + \sigma H_V} \right] U_1' \tag{3.5}$$

The trajectory of the visible brane **v** before the collision is $\bar{R}_v(\tau) = const$ and its expansion rate is, correspondingly, $H_v = 0$. The matching condition between regions **2** and **3** is obtained from (3.5) by taking $H_V \to 0$

$$U_3' = U_2' \,. \tag{3.6}$$

Finally, the trajectory of the brane **d** which is the mirror copy of the brane **b** is $\bar{R}_v(\tau)$. The values of parameter σ in the regions **2** and **3** are $\sigma_{2,3} = -\sigma_1 = \sigma$. From the matching condition (2.21) we find

$$U_4' = \left[\frac{F_2(\sqrt{H_b^2 + F_1/R^2} - \sigma H_b)}{F_1(\sqrt{H_b^2 + F_2/R^2} - \sigma H_b)} \right] U_3' \,. \tag{3.7}$$

Combining (3.4)–(3.7) we find

$$\frac{(\sqrt{H_V^2 + F_1/R^2} + \sigma H_V)(\sqrt{H_b^2 + F_1/R^2} - \sigma H_b)(\sqrt{H_b^2 + F_2/R^2} + \sigma H_b)}{(\sqrt{H_V^2 + F_1/R^2} - \sigma H_V)(\sqrt{H_b^2 + F_2/R^2} - \sigma H_b)(\sqrt{H_b^2 + F_1/R^2} + \sigma H_b)} = 1 \,. \tag{3.8}$$

In order to simplify the above expression, let us suppose that H_b^2, H_V^2 are small compared to the $F_1/R^2, F_2/R^2$. Then in the first approximation we get

$$H_V \approx H_b \left(1 - \sqrt{\frac{F_1}{F_2}} \right) \,. \tag{3.9}$$

The consistency condition (3.8) or (3.9) can be treated as a sort of energy conservation law for the brane collision. Indeed, the expansion rate H is related to the energy density ϵ on the brane through the five-dimensional analog of Friedman equation [6]

$$\sigma_r \sqrt{H^2 + F_r(R)/R^2} - \sigma_l \sqrt{H^2 + F_l(R)/R^2} = \epsilon \,, \tag{3.10}$$

where the indexes l, r refer to the regions on the left and on the right from the brane. For the visible brane **V** $\sigma_r = -\sigma_l = \sigma$ and $F_l = F_r = F_1$. The energy density ϵ consists of the brane tension λ_V and energy density of radiation ϵ_{rad} produced at the moment of collision. For $H_V^2 \ll F_1/R^2$ and $M_1 \ll \Lambda_1$ we find

$$\epsilon_{rad} \approx 2\sigma\sqrt{\Lambda_1} \left(1 + \frac{H_V^2}{2\Lambda_1} - \frac{M_1}{2\Lambda_1 R^4} \right) - \lambda_V \,. \tag{3.11}$$

If the tension of the visible brane is fine tuned to the bulk cosmological constant $\lambda_V = 2\sigma\sqrt{\Lambda_1}$ then

$$\epsilon_{rad} \approx \sigma \left(\frac{H_V^2}{\sqrt{\Lambda_1}} - \frac{M_1}{\sqrt{\Lambda_1} R^4} \right) \,. \tag{3.12}$$

If we take the visible brane of positive tension, $\sigma = 1$, as it is in pyrotechnic scenario [2] and the bulk metric in the region **1** to be anti-deSitter, $M_1 = 0$, we find from (3.9) and (3.12) that energy density of matter produced on the brane after the collision is

$$\epsilon_{\rm rad} \approx H_b^2 \left(\frac{1}{\sqrt{\Lambda_1}} - \frac{1}{\sqrt{\Lambda_2}} \right) . \tag{3.13}$$

And the initial expansion rate of the brane **V** is

$$H_V \approx H_b \left(1 - \sqrt{\frac{\Lambda_1}{\Lambda_2}} \right) . \tag{3.14}$$

Thus, if $\Lambda_2 > \Lambda_1$ some positive energy density is produced in the universe residing on the visible brane and the universe starts to expand (since the expansion rate of the bulk brane H_b was positive at the moment of collision).

If we consider the visible brane of negative tension, $\sigma = -1$, as it is in original ekpyrotic model of [1] and, for example choose $M_1 \neq 0$ in the region **1** we find

$$\begin{aligned} \epsilon_{\rm rad} &\approx \frac{M_1}{\sqrt{\Lambda_1}} \\ H_V &\approx H_b \frac{M_1}{2\sqrt{\Lambda_1}} \end{aligned} \tag{3.15}$$

right after the collision. We see that if the energy density of radiation produced on the visible brane is positive, the brane starts to collapse rather then expand since $H_b < 0$ at the moment of collision. This difficulty was discussed in [2].

4. Conclusion

We have derived a conservation law valid in brane collisions in five dimensional anti-deSitter space. Using this conservation law we have analyzed a particular situation when a brane sitting at an orbifold-fixed point in five-dimensional anti-deSitter space is hit by a bulk brane. We have found the energy density of radiation, ϵ_{rad}, produced on the orbifold-fixed brane. It is expressed through the expansion rate H_b (or, equivalently, kinetic energy) of the bulk brane at the moment of collision (3.13). We have found that if the tension of the visible brane is positive, the universe can start expansion after the collision, while in the case when the visible brane has negative tension the universe will start to collapse.

The method described in section 2 can be implemented for different background metrics. For example, we can develop a procedure, similar to the one used in section 3 for the bulk metric of the form

$$ds^2 = -D(y)dT^2 + B^2 D^4(y) dy^2 + A^2 D(y) \left(\sum_{k=1}^{3} dx_k^2 \right) , \tag{4.1}$$

where $D(y)$ is a given function, as it is in heterotic M-theory motivated Ansatz of ekpyrotic/pyrotechnik models.

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Since the conservation law (3.8) enables us to relate energy density on the visible brane after the collision to the energy density of the bulk brane before the collision, it is useful for the analysis of controversial question of behavior of density perturbations in the ekpyrotic models [1, 2, 7, 8, 9]. It is doubtful that this question can be resolved within an effective four-dimensional theory, since not only the equation of state of matter on the visible brane jumps at the moment of collision, but the energy density of matter itself. This is quite different from what is usually considered in discussion of matching the perturbations at the transitions between different epochs in the conventional four-dimensional cosmology.

Acknowledgments

I would like to thank the organizers of M-theory Cosmology conference in Cambridge during which some of this work was done. I am also grateful to V.Mukhanov and S.Solodukhin for discussions. This work was supported by SFB 375 der Deutsche Forschungsgemeinschaft.

A. Collision of light-like branes (figure 2)

The trajectory of the brane $\mathbf{b}_1$ in the coordinates (U_1, V_1) in the region **1** is given by equation

$$\Sigma_1^+ = V_1 - \alpha = 0\,, \tag{A.1}$$

where α is a constant. The same trajectory in the coordinates (U_2, V_2) in the region **2** is

$$\Sigma_2^- = V_2 - \beta = 0\,. \tag{A.2}$$

From the matching condition (2.6) we find a relation between the coordinates U_1 and U_2

$$\Lambda_1(\alpha - U_1) = \Lambda_2(\beta - U_2)\,. \tag{A.3}$$

Differentiating this equation with respect to u gives

$$U_2' = \frac{\Lambda_1}{\Lambda_2} U_1'\,. \tag{A.4}$$

Using the matching condition (2.5) we get also relation between V_1' and V_2'

$$V_2' = V_1'\,. \tag{A.5}$$

By the same way of reasoning applied to the boundary between regions **2** and **3** we get relations

$$U_3' = U_2'\,; \qquad V_3' = \frac{\Lambda_2}{\Lambda_3} V_2'\,. \tag{A.6}$$

For the boundary between regions **3** and **4** we get

$$U_4' = \frac{\Lambda_3}{\Lambda_4} U_3' \, ; \qquad V_4' = V_3' \tag{A.7}$$

and for the boundary between **4** and **1** we obtain

$$U_1' = U_4' \, ; \qquad V_1' = \frac{\Lambda_4}{\Lambda_1} V_4' \, . \tag{A.8}$$

Combining (A.4)–(A.8) we find the consistency condition (see (2.11))

$$\Lambda_1 \Lambda_3 = \Lambda_2 \Lambda_4 \, . \tag{A.9}$$

References

[1] J. Khoury, B.A. Ovrut, P.J. Steinhardt and N. Turok, *The ekpyrotic universe: colliding branes and the origin of the hot big bang*, `hep-th/0103239`; *A brief comment on 'the pyrotechnic universe'*, `hep-th/0105212`; *Density perturbations in the ekpyrotic scenario*, `hep-th/0109050`.

[2] R. Kallosh, L. Kofman and A.D. Linde, *Pyrotechnic universe*, `hep-th/0104073`.

[3] C.P. Burgess, M. Majumdar, D. Nolte, F. Quevedo, G. Rajesh and R.J. Zhang, *The inflationary brane-antibrane universe*, *J. High Energy Phys.* **07** (2001) 047 [`hep-th/0105204`].

[4] M. Bucher, *A braneworld universe from colliding bubbles*, `hep-th/0107148`.

[5] T. Dray and G. 't Hooft, *The effect of spherical shells of matter on the Schwarzschild black hole*, *Comm. Math. Phys.* **99** (1985) 613; *The gravitational effect of colliding planar shells of matter*, *Class. and Quant. Grav.* **3** (1986) 825.

[6] A.Y. Neronov, *Cosmological models with one extra dimension*, *Phys. Lett.* **B 472** (2000) 273 [`gr-qc/9911122`].

[7] D.H. Lyth, *The primordial curvature perturbation in the ekpyrotic universe*, `hep-ph/0106153`.

[8] R. Brandenberger and F. Finelli, *On the spectrum of fluctuations in an effective field theory of the ekpyrotic universe*, `hep-th/0109004`.

[9] J. Khoury, B.A. Ovrut, N. Seiberg, P.J. Steinhardt and N. Turok, *From big crunch to big bang*, `hep-th/0108187`.